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SOLUTIONS TO HEAT-CONDUCTION PROBLEMS BY A MIXED METHOD

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Abstract — A mixed formulation for the three-dimensional analysis of steady heat-conduction problems is presented. The method developed permits prescription of the boundary conditions in terms of either the temperature or the heat flux, or a combination of both. The general expressions for the temperature and the heat fluxes are derived in the form of a series in powers of the linear partial differential operators which operate on a set of initial functions to be determined by the prescribed boundary conditions. One of the important features of this method is its ability in providing approximate solutions in closed form. In order to illustrate the procedure some representative example problems with respect to thick plates have been solved, the corresponding solutions leading to results of engineering accuracy.

NOMENCLATURE

x, y, z,	rectangular cartesian co-ordinates;
$2l, 2h$,	length, and thickness of the plate (also $2l = L$);
$T(x, y, z)$,	temperature field;
q_x, q_y, q_z,	heat flow components, and $q = q_z$;
k,	thermal conductivity;
u''',	rate of heat generated in the body;
T_0, q_0,	values of T and q at the reference plane, $z = 0$;
$L_{TT}, L_{Tq}, L_{qT}, L_{qq}$,	linear partial differential operators;
B_r,	Biot number defined in equation (50);
H_r,	radiative boundary conductance.

Greek symbols

α, β, ξ,	differential operators, $= \partial/\partial x, \partial/\partial y, \partial/\partial z$ respectively;
γ^2,	$\alpha^2 + \beta^2 = \nabla^2$.

INTRODUCTION

SOLUTIONS to steady heat-conduction problems can be obtained by using various methods currently available in literature, for instance see references [1, 2]. The more widely used methods among these are those of the separation of variables which provide exact solutions, and the variational methods, in conjunction with either Ritz or Kontrovich procedure, which furnish approximate solutions. In all these methods the unknown independent quantity to be determined is the temperature field, and the associated boundary conditions are either the temperature or its first derivatives, or a combination of both. However, a mixed method, which involves the determination of both temperature and heat flux as independent quantities in association with a mixed form of boundary conditions expressed in terms of these quantities, has so far not been published.

In this paper a mixed method is presented for carrying out the three-dimensional analysis of steady heat-conduction problems. The proposed method is analogous to that employed in the study of elasticity problems, described earlier by Vlassov [3] and later applied to the analysis of thick plates by Iyengar *et al.* [4]. The differences as observed in the present formulation and the nature of the solutions obtained arise essentially from its application to the study of problems of a different branch of science. The method developed gives complete choice in prescribing the boundary conditions in terms of either the temperature or heat flux or a combination of both. The general expressions for the temperature field and the heat flux are derived in a series form in powers of the co-ordinate in the thickness direction and also in powers of the linear differential operators with respect to the other two co-ordinates, which operate on a set of initial functions to be evaluated by the prescribed boundary conditions or, alternately, these quantities can also be expressed in a transcendental form in terms of trigonometric functions containing the coordinate in the thickness direction and the differential operators operating on the initial functions. This procedure reduces the three-dimensional formulation to a two-dimensional one containing the derivatives operating on the initial functions, thereby decreasing the complexity involved. The procedure to be followed in arriving at the solution is illustrated by applying it to the analysis of thick plates (or slabs); some representative example problems are considered for this purpose. The proposed formulation can provide exact solutions which can be obtained by retaining all terms in the series expansions, referred to above. The corresponding solutions in respect of certain class of problems are found to be identical to those obtained by the method of separation of variables. However, the forte of the method is its ability in providing approximate solutions in closed form.

From the stress analysis point of view it may be desirable, and also usually adequate, to retain only a finite number of terms in the series, which may lead to approximate solutions to the problems with engineering accuracy. These approximate solutions are, in many cases, found to be compatible with the finite number of terms correspondingly retained in the formulation of the associated thermal stress problems in which some of these terms will be present. Some of the results presented here have been used by the author in the analysis of thermal stress problems associated with thick plates [5]. A question arises as to the utility of approximate solutions when powerful numerical techniques are available which can be applied to problems associated with complex geometries and boundary conditions. In defence of approximate solutions it must be stated that frequently an engineer needs quick, though approximate, results for an immediate task on hand, for instance, when rapid estimates are desired or when analytical temperature expressions are needed for further calculations of thermal stresses, or when the general physical aspects of the solutions are to be emphasized. Therefore, such methods appear to fulfill a useful practical role, occupying a middle ground between exact solutions and purely numerical analyses. The proposed formulation can be applied to practical problems associated with complex geometries and boundary conditions, for instance, by using the least squares point matching technique in conjunction with a polar coordinate system. The least squares point matching technique has been applied to the analyses of temperature and thermal stress problems of thin rectangular plates with cutouts by using a polar coordinate system [6, 7]. The procedure to be followed in the analyses of thick plates is similar.

In order to illustrate the procedure three typical example problems have been solved and the results compared with those of the exact method, where possible. The comparison study has yielded interesting results which demonstrate the versatility of the proposed procedure.

FORMULATION OF THE PROBLEM

The governing equation for steady heat conduction in a solid is given as:

$$-\frac{\partial}{\partial x}q_x - \frac{\partial}{\partial y}q_y - \frac{\partial}{\partial z}q_z + u''' = 0 \tag{1}$$

and the associated Fourier's law of heat conduction for isotropic solids may be expressed as:

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y} \tag{2}$$

$$q_z = -k\frac{\partial T}{\partial z}. \tag{3}$$

In equations (1)–(3), q_x, q_y, q_z and T are heat flow components and temperature field respectively. k is the thermal conductivity and u''' the rate of heat generated in the body.

In carrying out further formulation the following symbols will be used:

$$\alpha = \partial/\partial x, \quad \beta = \partial/\partial y,$$
$$\xi = \partial/\partial z, \quad q_z = q,$$
$$\gamma^2 = \alpha^2 + \beta^2 = \nabla^2. \tag{4}$$

Substitution of equations (2) and (4) in equations (1) and (3) leads to the following expressions:

$$\xi q = k\gamma^2 T + u''' \tag{5}$$

$$\xi T = -q/k. \tag{6}$$

The heat conduction problem is completely defined by equations (5) and (6).

The general solution of equations (5) and (6) can now be expressed in terms of the unknown initial functions $T_0(x, y)$ and $q_0(x, y)$ on the reference plane; in other words T_0 and q_0 are the values of $T(x, y, z)$ and $q(x, y, z)$, respectively, evaluated at $z = 0$. This can be carried out by expanding the general solution of (5) and (6) in MacLaurin series as

$$T(x, y, z) = T_0(x, y) + z[\xi T]_{z=0} + \frac{z^2}{2!}[\xi^2 T]_{z=0} + \ldots \tag{7}$$

$$q(x, y, z) = q_0(x, y) + z[\xi q]_{z=0} + \frac{z^2}{2!}[\xi^2 q]_{z=0} + \ldots \tag{8}$$

The derivatives appearing in equations (7) and (8) can be readily derived from equations (5) and (6). Following the elimination of these derivatives the functions T and q can be completely expressed in terms of the initial functions T_0 and q_0 and their partial derivatives with respect to x and y as

$$T(x, y, z) = L_{TT}(z, \gamma)\cdot T_0 + L_{Tq}(z, \gamma)\cdot q_0 - \frac{z^2}{2}\frac{u'''}{k} \tag{9}$$

$$q(x, y, z) = L_{qT}(z, \gamma)\cdot T_0 + L_{qq}(z, \gamma)\cdot q_0 \tag{10}$$

where

$$L_{TT} = L_{qq} = \cos(\gamma z), \tag{11}$$
$$L_{Tq} = -\frac{1}{k\gamma}\sin(\gamma z),$$
$$L_{qT} = k\gamma\sin(\gamma z).$$

In equations (11) the linear differential operators L_{TT}, L_{qq}, ... etc., which can be expressed either in series form in powers of γz or in trigonometric functions of the argument γz, have been expressed in transcendental form containing the trigonometric functions, for convenience sake.

The heat fluxes, q_x, q_y can also be expressed in a similar manner as follows:

$$q_x = -k\alpha(\cos\gamma z)\cdot T_0 + \frac{\alpha}{\gamma}(\sin\gamma z)\cdot q_0 \tag{12}$$

$$q_y = -k\beta(\cos\gamma z)\cdot T_0 + \frac{\beta}{\gamma}(\sin\gamma z)\cdot q_0. \tag{13}$$

Using equations (9)–(13) two- or three-dimensional problems associated with various boundary conditions can be solved.

The method formulated will now be illustrated by applying it to heat-conduction problems of thick plates. Three representative example problems are solved for this purpose.

SAMPLE PROBLEMS

The general expressions for the temperature $T(x, y, z)$ and the heat flux $q(x, y, z)$ as given by equations (9) and (10) are used for forming the governing equations in the form appropriate to the nature of a particular problem for which the solution is desired. If no truncation is carried out in the series expansions of these equations for forming the governing equations, then the solutions to the latter are exact. The solutions can be assumed in the desired form such that they satisfy the prescribed boundary conditions exactly. This procedure was in fact employed for obtaining exact solutions in respect of three typical two- and three-dimensional problems and the solutions were found to be identical to those obtained by the method of separation of variables. The corresponding results are not presented here since the purpose of this paper is to demonstrate the versatility of the proposed method in obtaining approximate solutions in closed form.

On the other hand the governing equations formed with respect to a finite number of terms retained in the series, referred to above, lead to approximate solutions, the order of accuracy achieved depending on the number of terms considered in the series expansion. Using this procedure approximate solutions will now be obtained for three representative sample problems.

Example 1: Two-dimensional problem with both temperature and heat flux boundary conditions

The problem can be stated as follows (see Fig. 1)

$$-l \leqslant x \leqslant l, \quad -h \leqslant z \leqslant h, \quad -\infty \leqslant y \leqslant \infty$$

$$T = T(x, z)$$

$$T(-l, z) = T(l, z) = 0 \tag{14}$$

$$q(x, h) + Q(x) = 0; \; q(x, -h) = 0 \tag{15}$$

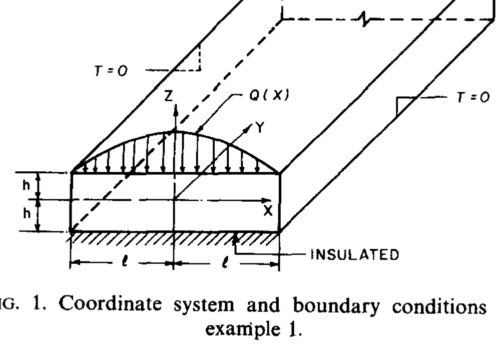

FIG. 1. Coordinate system and boundary conditions for example 1.

$$Q(x) = Q_0[1 - (x/l)^2]. \tag{16}$$

Solution. Using equations (10) and (11) and satisfying the boundary conditions (15) and also noting the two-dimensional character of the problem (i.e. $\gamma = \alpha$), the following equations are obtained:

$$\begin{aligned} L_{qT}(h) \cdot T_0 + L_{qq}(h) \cdot q_0 &= -Q(x) \\ -L_{qT}(h) \cdot T_0 + L_{qq}(h) \cdot q_0 &= 0. \end{aligned} \tag{17}$$

The second of equation (17) is satisfied by taking

$$T_0 = L_{qq}(h) \cdot \Phi, \; q_0 = L_{qT}(h) \cdot \Phi. \tag{18}$$

Substitution of equation (18) in the first of equation (17) leads to the following governing equation for Φ:

$$k\alpha \cdot (\sin 2\alpha h) \cdot \Phi = -Q(x). \tag{19}$$

The governing equation and the expressions for T and q_x as given by equations (19), (9) and (12) respectively, when expressed in series form, are given as:

$$k\alpha\left[2\alpha h - \frac{(2\alpha h)^3}{3!} + \frac{(2\alpha h)^5}{5!} - \ldots,\right]\Phi = -Q(x) \tag{20}$$

$$\begin{aligned} T = &\left[1 - \frac{z^2}{2!}\alpha^2 + \frac{z^4}{4!}\alpha^4 - \ldots,\right] \cdot \left[1 - \frac{h^2}{2!}\alpha^2 + \frac{h^4}{4!}\alpha^4 - \ldots,\right]\Phi \\ &+ \left[-z + \frac{z^3}{3!}\alpha^2 - \frac{z^5}{5!}\alpha^4 + \ldots,\right] \cdot \left[h\alpha^2 - \frac{h^3}{3!}\alpha^4 + \frac{h^5}{5!}\alpha^6 - \ldots\right]\Phi \end{aligned} \tag{21}$$

$$q_x = -k\alpha T. \tag{22}$$

In forming equations (20)–(22), equations (11) and (18) have been made use of. It can be observed that the coefficients of Φ as given in equation (20), have been obtained in series form in even powers, $2n$, of the differential operator α, ($n = 1, 2, \ldots \infty$). Solution to this equation can be obtained after retaining only a finite number of terms in the equation. The number of terms to be considered depends on the desired degree of accuracy and the type of the problem considered. As n increases from 1 to a finite number, say N, the number of boundary conditions associated with the problem that can be satisfied at each edge of the slab (in x direction) is equal to N and, therefore, the number of terms in the series, associated with powers of z in equations (21) and (22) that must be considered, is also N. Consequently, as the order of the governing equation increases the complexity of the problem also correspondingly increases.

In order to demonstrate the procedure, solutions corresponding to the first and second order approximations are obtained in the following manner. For the sake of convenience, $Q(x)$ is approximated by a polynomial function as given in equation (16).

First order approximation

The governing equation for this case as derived from equation (20) with only the first term retained, and the boundary condition (14), consistent with the symmetry of the problem about the z axis are obtained as:

$$\alpha^2\Phi = -Q_0(1 - x^2/l^2)/2kh \quad (23)$$

$$q_x(0, z) = T(l, z) = 0. \quad (24)$$

The series associated with powers of h and z in equations (20), (21) and (22) will be referred to as h-series and z-series, respectively. Consequent to the retention of one term only in the h-series of equation (20), it is necessary to retain one term only in each h-series of equations (21) and (22). However the order of equation (23), (which is $2n$, $n = 1$), requires retention of terms of the order of unity only in the z-series of equations (21) and (22).

Integration of equation (23) and satisfaction of the boundary condition (24) lead to the solution.

$$T(x, z)/\bar{Q}_0 = \left(\frac{5}{96}\right)\left(\frac{2l}{h}\right)\left[1 - \frac{6}{5}\left(\frac{x}{l}\right)^2 + \frac{1}{5}\left(\frac{x}{l}\right)^4\right] \quad (25)$$

where,

$$\bar{Q}_0 = 2Q_0 l/k. \quad (26)$$

Second order approximation

Considering two terms only in equation (20) a fourth order differential equation is obtained. The corresponding governing equation and its solution can be written as follows:

$$(\alpha^4 - H\alpha^2)\Phi = \left(\frac{3Q_0}{4kh^3}\right)[1 - (x/l)^2] \quad (27)$$

$$\Phi = A \cdot \cosh H^{1/2}x + B \cdot \sinh H^{1/2}x + cx + D + \left(\frac{2Q_0 l}{k}\right)\left(\frac{1}{96}\right)\left[\left(\frac{2l}{h}\right)\left(\frac{x}{l}\right)^4 + \left\{32\left(\frac{h}{2l}\right) - 6\left(\frac{2l}{h}\right)\right\}\cdot\left(\frac{x}{l}\right)^2\right] \quad (28)$$

where

$$H = \frac{3}{2h^2}.$$

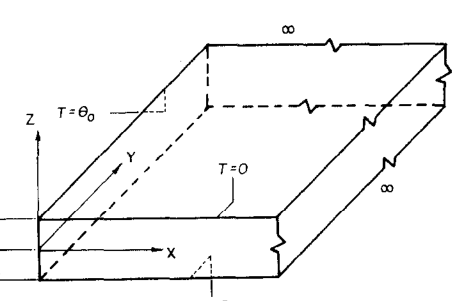

FIG. 2. Coordinate system and boundary conditions for example 2.

In equation (28) the first four terms are the homogeneous solutions and the remaining terms are particular integrals. The constants A, B, C and D are determined from the boundary conditions.

For this case two terms in each of the h-series and one term in each of the z-series in equations (21) and (22) must be considered, in view of the order of the governing equation being equal to $2n$ with $n = 2$.

Using equations (21), (22) and (28), the constants A, B, C and D can be determined from the boundary conditions (24). The final expression for T can be written as

$$\bar{T} = T/\bar{Q}_0 = [f_1 + f_2(x/l)^2 + f_3 (x/l)^4 + f_4 \cosh H^{1/2}x] + (z/h)[g_1 + g_2(x/l)^2 + g_3 \cosh H^{1/2}x] \quad (29)$$

where

$$f_1 = \{5/(96\bar{h})\} - \bar{h}/12 + (4/9)\bar{h}^3$$

$$f_2 = (\bar{h}/12) - (1/16\bar{h});\ \bar{h} = h/2l$$

$$f_3 = 1/(96\bar{h});\ f_4 = -(4/9)(\bar{h}^3/\cosh H^{1/2}l)$$

$$g_1 = -2\bar{h}^3 + \bar{h}/2;\ g_2 = -\bar{h}/2$$

$$g_3 = 2\bar{h}^3/\cosh H^{1/2}l.$$

It will be interesting to compare the above two approximate solutions with the exact solution, obtained by the method of separation of variables and associated with the boundary heat flux at $z = h$ represented by a single harmonic function (i.e. $Q(x) = Q_0 \sin \pi x/L$ with the origin located at the left edge of the slab, and $L = 2l$), very nearly equivalent to the distribution of the present problem.

The numerical values for the non-dimensionalized $\bar{T}$ at $x = z = 0$ for $h/2l = \frac{1}{6}$ corresponding to first and second approximation of the present example are found to be equal to 0.3125 and 0.3006, respectively. Comparing these values with the exact value of 0.2905, the percentage errors in respect of first and second approximation results are 7.5 and 3.46 respectively. Since the problem is asymmetric with respect to the X-axis, the solution corresponding to the first approximation, which is found to be symmetric about this axis, gives very inaccurate results for non-zero values of z. The solution obtained from the second order approximation provides reasonably good results for small values of $h/2l$. For large values of $h/2l$ approaching unity additional terms in equation (20) must be considered.

Example 2: Two-dimensional problem with temperature boundary conditions only

The problem may be stated as follows (see Fig. 2)

$$0 \leqslant x \leqslant \infty,\ -h \leqslant z \leqslant h,\ -\infty \leqslant y \leqslant \infty$$

$$T = T(x, z) \quad (30)$$

Boundary conditions:

$$T(x, -h) = T(x, h) = 0 \quad (31)$$

$$T(0, z) = \theta_0 \qquad (32)$$

Symmetry condition:

$$q_0 = 0. \qquad (33)$$

Solution. The general expression for the temperature as given by equation (9) subject to the symmetry condition (33) can be written as

$$T(x, z) = \left[1 - \frac{z^2}{2!}\alpha^2 + \frac{z^4}{4!}\alpha^4 - \ldots,\right] T_0. \qquad (34)$$

Following the satisfaction of the boundary conditions (31), equation (34) gives

$$\left[1 - \frac{h^2}{2!}\alpha^2 + \frac{h^4}{4!}\alpha^4 - \ldots,\right] T_0 = 0. \qquad (35)$$

Solutions to equation (35) will now be presented for two cases (cases 1 and 2) corresponding to two and three terms, respectively, retained in the above h-series; the governing equations for these two cases are given as

Case 1:

$$(\alpha^2 - 2/h^2)\, T_0 = 0 \qquad (36)$$

Case 2:

$$\left(\alpha^4 - \frac{12}{h^2}\alpha^2 + \frac{24}{h^4}\right) T_0 = 0. \qquad (37)$$

Solutions to the second and fourth order equations (36) and (37) which permit satisfaction of one and two boundary conditions only, respectively, at each boundary associated with the X axis can be written as

Case 1:

$$T_0(x) = E \cdot e^{-\beta x/h} + F \cdot e^{\beta x/h}. \qquad (38)$$

Case 2:

$$T_0(x) = A \cdot e^{-\lambda_1 x/h} + B \cdot e^{-\lambda_2 x/h} + C \cdot e^{\lambda_3 x/h} + D \cdot e^{\lambda_4 x/h}. \qquad (39)$$

In equations (38) and (39) A, B, C, D, E and F are arbitrary constants to be determined from the boundary conditions, and λ_1 to λ_4 and β are known constants. It may be pointed out here that the heat applied to the boundary at $x = 0$ with $T = \theta_0$ cannot be expected to have much influence at large distances from this boundary, since the slab is infinite in extent in the x direction; from this physical consideration the constants F, C and D can be taken to be equal to zero. Consideration of one term only for case 1 and two terms only for case 2 in the z-series of equation (34), leads to the determination of the remaining constants E, A and B. The final solutions to the problem can then be written as:

Case 1:

$$T/\theta_0 = (1 - z^2/h^2)\, e^{-\beta x/h} \qquad (40)$$

Case 2:

$$T/\theta_0 = \left(\frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2}\right) \cdot \left\{1 - \frac{z^2}{h^2} \cdot \frac{\lambda_1^2}{2} + \frac{z^4}{h^4} \cdot \frac{\lambda_1^4}{24}\right\} \cdot e^{-\lambda_1 x/h} - \left(\frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2}\right)\left\{1 - \frac{z^2}{h^2} \cdot \frac{\lambda_2^2}{2} + \frac{z^4}{h^4} \cdot \frac{\lambda_2^4}{24}\right\} \cdot e^{-\lambda_2 x/h} \qquad (41)$$

where

$$\beta = \sqrt{2}$$

$$\lambda_1 = (6 + 2\sqrt{3})^{1/2},\ \lambda_2 = (6 - 2\sqrt{3})^{1/2}.$$

It can be observed from equations (40) and (41) that the boundary conditions (31) are exactly satisfied. However, the boundary conditions (32) is not exactly satisfied since the number of terms considered in the z-series of equation (34) for satisfying this boundary condition is less by one than the corresponding number of terms retained in this series for forming the governing equations (36) and (37). This point will now be examined in greater detail.

The expressions for the temperature at the boundary $x = 0$ obtained from equations (40) and (41) and also the corresponding expression as given by the method of separation of variables, (see equation 4-71, p. 199 of [2]), are given as

Case 1: $$T/\theta_0 = 1 - (z/h)^2 \qquad (42)$$

Case 2: $$T/\theta_0 = 1 - (z/h)^4 \qquad (43)$$

Case 3: $$T/\theta_0 = (2/h) \sum_{n=0}^{\infty} \frac{(-1)^n}{\delta^n} \cos \delta_n z \qquad (44)$$

where $\delta_n = (2n + 1)\pi/2h$, $n = 0, 1, 2, \ldots$.

Equations (42) and (43) satisfy the boundary condition at $x = 0$, exactly at the point $z = 0$ and at points with $z \neq 0$ the errors are of the order of z^2 and z^4 for cases 1 and 2 respectively. Clearly case 2, in which the number of terms retained is greater by one than that for case 1, gives more accurate results than case 1. The accuracy of results can be increased by retaining more number of terms in the z-series of equation (34), which inevitably increases the complexity of the problem. It may be noted that case 3, which corresponds to the method of separation of variables, can satisfy the

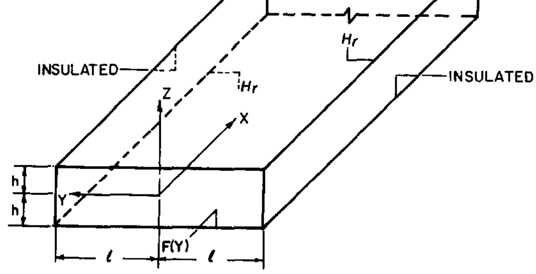

FIG. 3. Coordinate system and boundary conditions for example 3.

boundary condition at $x = 0$ only by considering sufficient number of terms in the Fourier cosine series; it is interesting to note that this method cannot satisfy the boundary condition exactly by retaining one term only even at $z = 0$, whereas the present method does, irrespective of the number of terms considered in the general expression for the temperature.

Example 3: Three dimensional problem with mixed boundary conditions

The problem may be stated as follows (see Fig. 3):

$T = T(x, y, z)$

$$0 \leqslant x \leqslant \infty, \quad -l \leqslant y \leqslant l, \quad -h \leqslant z \leqslant h$$

Boundary conditions:

$$q_y(x, -l, z) = q_y(x, l, z) = 0 \tag{45}$$

$$q(x, y, \pm h) - H_r[T(x, y, \pm h) - \zeta(x, y)] = 0 \tag{46}$$

$$T(0, y, z) = F(y) = \theta_m \cos m\pi y/l;$$

$$m = 0, 1, 2, \ldots \infty. \tag{47}$$

Symmetry condition:

$$q_0 = 0. \tag{48}$$

Equation (46) is a simple form of radiation boundary condition which is similar in form to the commonly employed convective boundary condition. In this equation H_r is the radiative boundary conductance, which is assumed to be a constant in the present problem, and $\zeta(x, y)$ is the temperature of the enclosure around the slab.

Substitution of equations (9), (10) and (48) in equation (46) leads to

$$\left(\frac{\gamma h}{B_r} \sin \gamma h - \cos \gamma h\right) T_0 = -\zeta(x, y) \tag{49}$$

where

$$B_r = hH_r/k. \tag{50}$$

Equation (49) is the governing equation whose solution can be determined for a given distribution of $\zeta(x, y)$. For the purpose of illustrating the procedure, ζ will be taken to be a constant. It is convenient to obtain the solution by considering ζ to be a reference temperature such that

$$T(x, y, z) = \bar{T}(x, y, z) + \zeta$$

and the modified problem associated with $\bar{T}(x, y, z)$ can then be stated as follows:

$$\bar{T} = \bar{T}(x, y, z) \tag{51}$$

Boundary conditions:

$$q(x, y, h) - H_r \bar{T}(x, y, h) = 0 \tag{52}$$

$$\bar{T}(0, y, z) = F'(y) = \bar{\theta}_m \cos m\pi y/l;$$

$$\bar{T}(\infty, y, z) = 0 \tag{53}$$

$$q_0 = 0. \tag{54}$$

The modified governing equation derived from equations (9) and (10) with T_0 replaced by $\bar{T}_0$ and from equations (52) and (54) can be expressed as

$$(\gamma h \sin \gamma h - B_r \cos \gamma h)\, \bar{T}_0 = 0. \tag{55}$$

Equation (55) can be expanded in series in even powers of γh. The accuracy of the solution to this equation is dependent on the number of terms retained in the series. For illustration the governing equation as derived from equation (55) with first three terms retained in the series will now be considered for obtaining the solution.

The corresponding governing equation and the associated solution satisfying the boundary conditions (53) are given as

Governing equation:

$$\left[\nabla^4 - \left(\frac{\beta_1}{\beta_2 h^2}\right)\nabla^2 + \frac{1}{\beta_2 h^4}\right]\bar{T}_0 = 0. \tag{56}$$

Solution:

$$\bar{T}/\bar{\theta}_m = \left(\frac{\delta_2^2}{\delta_2^2 - \delta_1^2}\right)\left[1 - \frac{\delta_1^2}{2}\left(\frac{z}{h}\right)^2 + \frac{\delta_1^4}{24}\left(\frac{z}{h}\right)^4\right] \times e^{-\lambda_m x/h} \cos \eta_m y - \left(\frac{\delta_1^2}{\delta_2^2 - \delta_1^2}\right) \times \left[1 - \frac{\delta_2^2}{2}\left(\frac{z}{h}\right)^2 + \frac{\delta_2^4}{24}\left(\frac{z}{h}\right)^4\right] e^{-g_m x/h} \cos \eta_m y \tag{57}$$

where

$$\lambda_m = [\delta_1^2 + (\eta_m h)^2]^{1/2}; \quad g_m = [\delta_2^2 + (\eta_m h)^2]^{1/2}$$

$$\delta_{1,2} = [\{\beta_1 \pm (\beta_1^2 - 4\beta_2)^{1/2}\}/2\beta_2]^{1/2} \tag{58}$$

$$\beta_1 = (1 + 2/B_r)/2; \quad \beta_2 = (1 + 4/B_r)/24$$

$$\eta_m = m\pi/l.$$

In the expression for δ_1 and δ_2 as given by equation (58) the positive and negative signs correspond to the subscripts 1 and 2 of δ respectively. It may be noted from the equation (57) that the boundary condition (52) is exactly satisfied while the first of boundary condition (53) (at $x = 0$) is exactly satisfied at the point $z = 0$ and at points with $z \neq 0$ the error is of the order

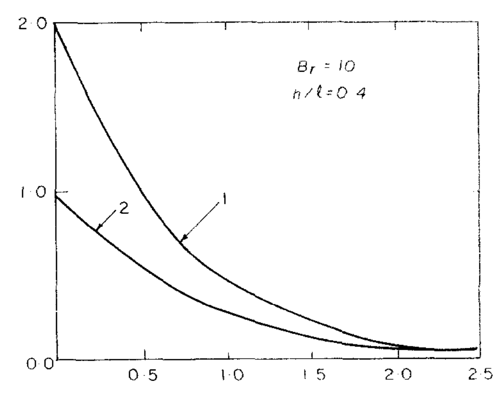

FIG. 4. Temperature distribution along the length of the plate at $z/h = 0$, 1: $y/l = 0$; 2: $y/l = 0.5$.

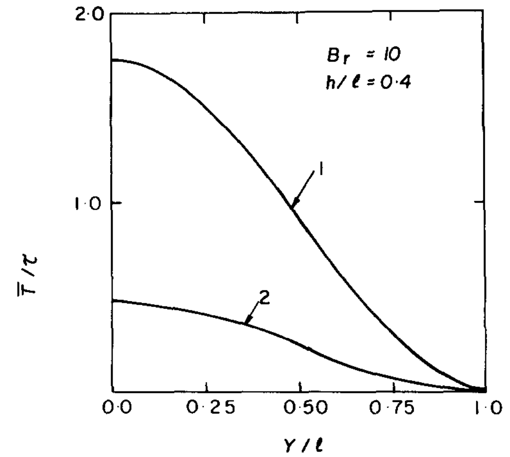

FIG. 5. Temperature distribution across the width of the plate at $x/h = 0.1$, 1: $z/h = 0$; 2: $z/h = 1$.

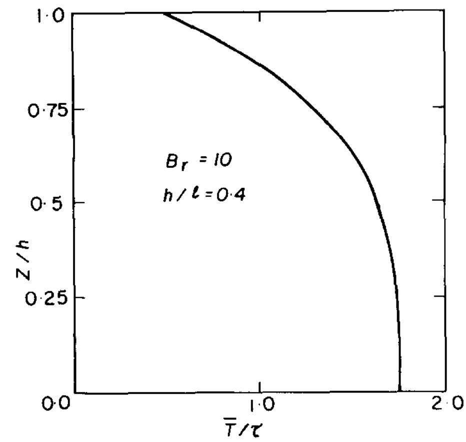

FIG. 6. Temperature distribution across the thickness of the plate at $x/h = 0.1$ and $y/l = 0$.

of z^4. This behaviour was also observed in the previous example, and the reason given there for this behaviour holds good for the present example also. The presence of the decaying exponential functions in the solution (57) suggests that the second of boundary condition (53) is also satisfied.

In order to provide a better insight into the nature of the general solution (57) a numerical example has been solved by taking $F'(y) = \tau(1 + \cos \pi y/l)$, (see Fig. 3), for $B_r = 10$ and the thickness ratio $h/l = 0.4$. Figures 4 and 5 display the corresponding temperature distributions at three different cross sections, while Fig. 6 shows the distribution in the thickness direction at $x/h = 0.1$ and $y/l = 0$. It is clear from Fig. 4 that the temperature distribution exhibits a decaying character with increases in x/h values as indicated by the presence of the exponential functions in the solution (57).

CONCLUSIONS

A mixed method for the three-dimensional analysis of steady heat conduction in solids was presented. One of the interesting features of the formulation was the flexibility offered in prescribing the boundary conditions in terms of either the temperature or the heat flux, or both. The forte of the method was its ability in providing approximate solutions in closed form. It was demonstrated through example problems that the approximate solutions would lead to results of engineering accuracy.

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SOLUTIONS DES PROBLEMES DE CONDUCTION THERMIQUE PAR UNE METHODE MIXTE

Résumé — On présente une méthode mixte pour l'analyse tridimensionnelle des problèmes de conduction thermique stationnaire. La méthode développée permet la traduction des conditions aux limites en terme soit de flux ou de température, soit des deux. Les expressions générales de la température et des flux sont obtenues sous la forme de séries puissances de l'opérateur linéaire qui opère sur un système de fonctions initiales à déterminer par les conditions aux limites données. Un des caractères importants de cette méthode est son aptitude à fournir des solutions approchées sous une forme analytique. Pour illustrer la procédure, on résout quelques problèmes représentatifs de plaques épaisses dont les solutions sont d'une précision intéressante pour l'ingénieur.

LÖSUNGEN VON WÄRMELEITUNGSPROBLEMEN DURCH EIN KOMBINIERTES VERFAHREN

Zusammenfassung—Es wird eine kombinierte Formulierung für die dreidimensionale Untersuchung von stationären Wärmeleitproblemen angegeben. Das entwickelte Verfahren gestattet Vorgaben der Randbedingungen entweder durch Angabe der Temperatur oder des Wärmestroms oder einer Kombination von beiden. Die allgemeinen Ausdrücke für die Temperatur und die Wärmeströme werden in Form einer Potenzreihe der linearen partiellen Differentialoperatoren angegeben, die von einem Satz von Anfangsfunktionen, bestimmt durch die vorgegebenen Randbedingungen, dargestellt werden. Eine der wichtigsten Eigenschaften dieses Verfahrens ist die Möglichkeit, Näherungslösungen in geschlossener Form anzugeben. Um die Vorgehensweise zu veranschaulichen, sind einige repräsentative beispielhafte Problem dicker Platten gelöst worden, bei denen die entsprechenden Lösungen zu Ergebnissen von ingenieurmäßiger Genauigkeit führen.

РЕШЕНИЕ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ КОМБИНИРОВАННЫМ МЕТОДОМ

Аннотация — Дана комбинированная формулировка стационарных задач теплопроводности. Предлагаемый метод позволяет выражать задаваемые граничные условия через температуру или плотность теплового потока, или через их сочетание. Получены общие выражения для температуры и плотности теплового потока в виде рядов по степеням линейных дифференциальных операторов, которые действуют на систему функций, определяемых с помощью задаваемых граничных условий. Одним из основных достоинств метода является то, что он позволяет получать приближенные решения в замкнутом виде. С целью иллюстрации метода дан пример решения нескольких характерных задач для пластин большой толщины. Точность полученных результатов соответствует требованиям, предъявляемым инженерными расчётами.